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# KOSZUL DUALITY COMPLEXES FOR THE COHOMOLOGY OF ITERATED LOOP SPACES OF SPHERES

by

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**Abstract.** — The goal of this article is to make explicit a structured complex computing the cohomology of a profinite completion of the  $n$ -fold loop space of a sphere of dimension  $d < n$ . Our construction involves: the free complete algebra in one variable associated to any fixed  $E_n$ -operad; an element in this free complete algebra determined by a morphism from the operad of  $L_\infty$ -algebras to an operadic suspension of our  $E_n$ -operad. As such, our complex is defined purely algebraically in terms of characteristic structures of  $E_n$ -operads.

We deduce our main theorem from several results obtained in a previous series of articles – namely: a connection between the cohomology of iterated loop spaces and the cohomology of algebras over  $E_n$ -operads, and a Koszul duality result for  $E_n$ -operads. We use notably that the Koszul duality of  $E_n$ -operads makes appear structure maps of the cochain algebra of spheres.

## Introduction

The thesis of J.-P. Serre, at the beginning of the fifties, introduced the idea of using the cohomology of loop spaces in topology. Topologists have quickly studied the cohomology of iterated loop spaces  $\Omega^n X$  which are spaces of maps  $f : S^n \rightarrow X$  from spheres  $S^n$ . The computation of the cohomology of  $\Omega^n X$  would make possible the determination of homotopy classes of these maps  $f : S^n \rightarrow X$  but, for the moment, this computation is well understood only when  $X$  is the  $n$ -fold suspension of a space  $X = \Sigma^n Z$ . Besides, we have a comprehensive knowledge of the homotopy classes of finite cell complexes in a few particular cases only (which do not include spheres) for which the homotopy vanishes in dimension  $n > 1$ .

Our works [7, 9] give an algebraic device to model iterated loop spaces  $\Omega^n X$  on all spaces  $X$  and for every value of  $n$ . The goal of this paper is to explain how to make explicit a structured complex computing the cohomology of a profinite completion of

not restricted to  $n$ -fold  
suspensions

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the iterated loop spaces  $\Omega^n S^{n-m}$  from this model, by putting together results of [7, 9] and of another article [10]. The obtained complex is defined purely algebraically, in terms of the internal structure of certain operads, and inherits nice operadic structures from which we hope to get information of new nature on the topology of  $\Omega^n S^{n-m}$ .

The  $E_n$ -operads give the core structures of our construction. For our purpose, we consider  $E_n$ -operads in the category of differential graded modules (dg-modules for short) and we use the noun  $E_n$ -algebra to refer to any category of algebras associated to an  $E_n$ -operad in that ambient category. In that context, the  $E_n$ -operads refer to the homotopy type of operads which are weakly equivalent to the chain operad of Boardman-Vogt little  $n$ -cubes.

Most models of  $E_n$ -operads come in nested sequences

$$(*) \quad \mathbf{E}_1 \subset \cdots \subset \mathbf{E}_{n-1} \subset \mathbf{E}_n \subset \cdots$$

such that  $\mathbf{E}_\infty = \operatorname{colim}_n \mathbf{E}_n$  is an  $E_\infty$  operad, an operad weakly equivalent to the operad  $\mathbf{C}$  of associative and commutative algebras. The  $E_1$ -operads, which form the initial terms of nested sequences of the form  $(*)$ , are weakly equivalent to the operad  $\mathbf{A}$  of associative non-commutative algebras. Intuitively, an  $E_1$ -algebra structure is interpreted as a structure which is homotopy associative in the strong sense but non-commutative, while an  $E_\infty$ -algebra is fully homotopy (associative and) commutative. The  $E_n$ -algebra structures, which are intermediate between  $E_1$  and  $E_\infty$ , are interpreted as structures which are homotopy associative in the strong sense and homotopy commutative up to some degree. A commutative algebra naturally inherits an  $E_n$ -algebra structure, for all  $n = 1, 2, \dots, \infty$ .

Throughout this paper, a space means a simplicial set and we associate to a simplicial set  $X$  the standard normalized complex  $N^*(X)$  with coefficients in the ground ring  $\mathbb{k}$  of our base category of dg-modules. As we deal with pointed spaces, we use the reduced complex  $\bar{N}^*(X) = \ker(N^*(X) \rightarrow \mathbb{k})$  in our constructions rather than the whole complex  $N^*(X)$ . To handle structures without unit (like these reduced complexes  $\bar{N}^*(X)$ ), we also remove 0-ary operations from our operads.

The cochain algebras  $\bar{N}^*(X)$  inherit an  $E_\infty$ -algebra structure by [14], for which we have an explicit construction (see [2, 18]) giving an action of certain  $E_\infty$ -operads equipped with a filtration of the form  $(*)$  on  $\bar{N}^*(X)$ . In that context, we immediately see that the  $E_\infty$ -algebra  $\bar{N}^*(X)$  inherits an  $E_n$ -algebra structure for all  $n$ .

Now a natural homology theory  $H_*^{E_n}(A)$  is associated to any category of algebras over an  $E_n$ -operad. For a cochain algebra  $A = \bar{N}^*(X)$ , the theorems of [7, 9], put together, imply that the homology  $H_*^{E_n}(\bar{N}^*(X))$  determines the cohomology of a profinite completion of the  $n$ -fold loop space  $\Omega^n X$ .

But we observe in [10] that  $E_n$ -operads are in a sense Koszul and we prove, using a good  $E_n$ -operad  $\mathbf{E}_n$ , that a cofibrant model of  $E_n$ -operad is yielded by an operadic cobar construction  $B^c(\mathbf{D}_n)$ , where  $\mathbf{D}_n = \Lambda^{-n} \mathbf{E}_n^\vee$  is an operadic desuspension of the dual cooperad of  $\mathbf{E}_n$  in dg-modules. This result implies that the homology  $H_*^{E_n}(A)$  of any  $E_n$ -algebra can be computed by a nice structured complex of the form  $C_*^{E_n}(A) = (\mathbf{D}_n(A), \partial)$ , where  $\mathbf{D}_n(A)$  is the connected  $\mathbf{D}_n$ -cofree coalgebra on  $A$

and  $\partial$  is a twisting coderivation determined by the  $E_n$ -algebra structure of  $A$  (this construction is reviewed in §1).

The goal of this paper is to determine the differential of this complex  $C_*^{E_n}(A)$  for the cochain algebra of a sphere  $A = \bar{N}^*(S^{n-m})$ , where  $0 \leq m \leq n$ . In [10], we observe that the action of  $B^c(D_n)$  on  $\bar{N}^*(S^{n-m})$ , by way of the morphism  $B^c(D_n) \xrightarrow{\sim} E_n \subset E_\infty$ , is determined by a composite of:

- (a) the image under the cobar construction  $B^c(-)$  of a morphism  $\iota^* : D_n \rightarrow \Lambda^{m-n} D_m$  dual in dg-modules to the embedding  $\iota : E_m \rightarrow E_n$ ;
- (b) the operadic suspension of an augmentation morphism  $\phi_m : B^c(D_m) \rightarrow \mathbb{C}$  which, by operadic bar duality, is equivalent to a morphism  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^m E_m$  on the suspension of the operad  $L_\infty$  associated to the standard category of  $L_\infty$ -algebras, the classical version of homotopy Lie algebra structure.

The existence of a morphism  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^m E_m$  reflects the existence of a restriction functor from  $E_m$ -algebras to  $L_\infty$ -algebras and has been used in the literature in the case  $m = 2$ , in conjunction with the Deligne conjecture, to determine the homotopy type of certain deformation complexes (see for instance [17]). In [11], we prove that these morphisms  $\phi_m^\sharp$  are unique up to a right homotopy in the category of operads, and hence, are characteristic of the structure of an  $E_m$ -operad.

We give a conceptual definition of the complex  $C_*^{E_n}(\bar{N}^*(S^{n-m}))$  in terms of the embedding  $\iota : E_m \rightarrow E_n$ , of the morphism  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^m E_m$  and of the composition structure of the operad  $E_n$ . We will obtain a result of a particularly simple form also because the reduced normalized complex  $\bar{N}^*(S^{n-m})$  is a free  $\mathbb{k}$ -module of rank one. We give all details in §3. For this introduction, we are simply going to state the final outcome of our construction: the application of our result to iterated loop space cohomology as it arises from [7, 9].

We dualize the construction to ease the formulation of our result. We form, for any  $n \geq m$ , the free complete  $E_n$ -algebra  $\hat{E}_n(x)$  on one generator  $x$  of degree  $-m$ . The elements of this algebra  $\hat{E}_m(x)$  are just formal power series  $\sum_{r=1}^{\infty} \xi_r(x, \dots, x)$  where  $\xi_r$  is an  $r$ -ary operation of the operad  $E_m$ . Our result makes appear a composition product  $\circ : \hat{E}_n(x) \otimes \hat{E}_n(x) \rightarrow \hat{E}_n(x)$ , borrowed ~~to~~ from [4], defined termwise by sums of operadic substitution operations

$$\xi_r(x, \dots, x) \circ \zeta_s(x, \dots, x) = \sum_{i=1}^r \xi_r(x, \dots, \zeta_s(x, \dots, x), \dots, x)$$

running over all entries of  $\xi_r(x, \dots, x)$ . We observe that the existence of a morphism  $\phi^\sharp : \Lambda L_\infty \rightarrow \Lambda^m E_m$  is formally equivalent to the existence of an element  $\omega_m \in \hat{E}_m(x)$ , of degree  $-1 - m$ , satisfying the relation  $\delta(\omega_m) = \omega_m \circ \omega_m$ , where  $\delta$  refers to the canonical differential of  $\hat{E}_m(x)$ . We embed this element  $\omega_m$  into  $\hat{E}_n(x)$ , for any  $n \geq m$ . We have then:

**Theorem.** — We consider the twisted complex  $(\hat{E}_n(x), \partial_m)$  defined by adding the map  $\partial_m(\xi) = \xi \circ \omega_m$  to the natural differential of the free complete  $E_n$ -algebra  $\hat{E}_n(x)$ . When

the structure defined by a whole sequence of Lie brackets  $[-, \dots, -]_r$ ,  $r = 2, 3, \dots$ , satisfying higher homotopy Jacobi relations (see Hinich, V. and Schechtman, V.: Homotopy Lie algebras, In I. M. Gelfand Seminar, Adv. Soviet Math. 16(2), Amer. Math. Soc., Providence, RI, 1993, pp. 1–28).

the ground ring is  $\mathbb{Z}$ , we have an identity

$$H^*(\widehat{\Omega^n S^{n-m}}) = H_*(\widehat{E}_n(x)^\vee, \partial_m^\vee)$$

between the continuous cohomology of the profinite completion of  $\Omega^n S^{n-m}$  and the homology of the dual complex of  $(\widehat{E}_n(x), \partial_m)$  in  $\mathbb{Z}$ -modules (we take the continuous dual with respect to the topology of power series).

The profinite completion of the theorem is defined by the limit of a diagram  $\Omega^n S_\alpha^{n-m}$  where  $S_\alpha^{n-m}$  ranges over homotopy finite approximations of  $S^{n-m}$  (see [19]). This formulation is slightly abusive because the profinite completion of  $\Omega^n S^{n-m}$  would be the limit of a diagram of homotopy finite approximations of the space  $\Omega^n S^{n-m}$  itself. ~~Nevertheless results of [20] imply that these limits coincide.~~ → (not exactly, see [7]) The theorem has an  $\mathbb{F}_p$ -version where the profinite completion of  $\Omega^n S^{n-m}$  is replaced by a  $p$ -profinite approximation, the limit of a diagram  $\Omega^n S_\alpha^{n-m}$  where  $S_\alpha^{n-m}$  now ranges over homotopy  $p$ -finite approximations of  $S^{n-m}$  (see again [19]). The topological results of [7, 9] are stated in the  $\mathbb{F}_p$ -setting, but the arguments work ~~same~~ over  $\mathbb{Z}$ . the same way

The above theorem is not intended for computational purposes and we should not undertake any computation with it – the iterated bar complexes of [9], which are equipped with nice filtrations, seem better suited for such an undertaking. The actual goal of this article, with this theorem, is to make explicit the connection between the cohomology of iterated loop spaces of spheres and the homotopical structures of  $E_n$ -operads. In subsequent work, we plan to apply the theorem to gain qualitative information on the structure of iterated loop space cohomology by using connections, explained in [17], between  $E_n$ -operads and quantum field theory.

In §1, we briefly review the definition of the homology  $H_*^P(A)$  associated to an operad  $P$ , the applications of the bar duality of operads to the definition of complexes  $C_*^P(A)$  computing  $H_*^P(A)$ , and we explain how to apply this construction to  $E_n$ -operads with the result of [10]. In §2, we explain the definition of the free complete algebras of the form  $\widehat{E}_n(x)$  and their duality connection with cofree coalgebras  $D(C)$  on a dg-module  $C$  of rank one. In §3, we study the applications of operadic bar duality to the cochain algebra of spheres  $\bar{N}^*(S^{n-m})$  and we determine the form of the complex  $C_*^{E_n}(\bar{N}^*(S^{n-m}))$  from which we obtain the outcome of our theorem.

In §1 and 3, we deal with a particular  $E_n$ -operad, already used in the proof of [10], which arises from a certain filtration of an  $E_\infty$ -operad introduced by Barratt-Eccles in [1]. In §4, we explain that the homotopy type of the complex  $(\widehat{E}_n(x)^\vee, \partial_m^\vee)$  which occurs in our main theorem does not depend on the choice of the  $E_n$ -operad  $E_n$  and of the element  $\omega_m \in \widehat{E}_n(x)$  associated to a morphism  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^n E_n$ .

To begin with, we review the background of our constructions.

### Background and overall conventions

We are only going to give a brief survey of this background since our main purpose is to explain our conventions. We refer to [6] for a more detailed account with the same overall conventions as in the present article.

To begin with, we explicitly specify which base category of dg-modules is considered throughout the article.

*0.1. The category of dg-modules.* — The ground ring, fixed once and for all, is denoted by  $\mathbb{k}$ . For us a dg-module refers to a  $\mathbb{Z}$ -graded  $\mathbb{k}$ -module  $C$  equipped with an internal differential  $\delta : C \rightarrow C$  that decreases the degree by one, and we take this category of dg-modules as base category  $\mathcal{C}$ .

The category of dg-modules is equipped with its standard tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  together with the symmetry isomorphism  $\tau : C \otimes D \rightarrow D \otimes C$  involving a sign. This sign is determined by the standard sign convention of differential graded algebra, which we follow. The notation  $\pm$  is used to represent any sign arising from an application of this convention. The ground ring  $\mathbb{k}$ , viewed as a dg-module of rank 1 concentrated in degree 0, defines the unit object for the tensor product of dg-modules.

The morphism sets of any category  $\mathcal{A}$  are denoted by  $\text{Mor}_{\mathcal{A}}(A, B)$ . The morphisms of dg-modules are the degree and differential preserving morphisms of  $\mathbb{k}$ -modules  $f : C \rightarrow D$ . The category of dg-modules comes also equipped with internal hom-objects  $\text{Hom}_{\mathcal{C}}(C, D)$  characterized by the adjunction relation  $\text{Mor}_{\mathcal{C}}(K \otimes C, D) = \text{Mor}_{\mathcal{C}}(K, \text{Hom}_{\mathcal{C}}(C, D))$  with respect to the tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ . Recall briefly that a homogeneous element of  $\text{Hom}_{\mathcal{C}}(C, D)$  is just a morphism of  $\mathbb{k}$ -modules  $f : C \rightarrow D$  raising degrees by  $d = \deg(f)$ . The differential of a homomorphism  $f \in \text{Hom}_{\mathcal{C}}(C, D)$  is defined by the (graded) commutator  $\delta(f) = \delta \cdot f - \pm f \cdot \delta$ , where we consider the internal differentials of  $C$  and  $D$ . The elements of the dg-hom  $\text{Hom}_{\mathcal{C}}(C, D)$  are called homomorphisms to be distinguished from the actual morphisms of dg-modules, the elements of the morphism set  $\text{Mor}_{\mathcal{C}}(C, D)$ .

The dual of a dg-module  $C$  is the dg-module such that  $C^{\vee} = \text{Hom}_{\mathcal{C}}(C, \mathbb{k})$ , where we again view the ground ring  $\mathbb{k}$  as a dg-module of rank 1 concentrated in degree 0.

*0.2. Operads and cooperads.* — We always consider operads (and cooperads) in the category of dg-modules. We adopt the notation  $\mathcal{O}$  for the whole category of operads in dg-modules. We use the notation  $\mathbf{l}$  for the initial object of the category of operads which reduces to the ground ring  $\mathbf{l}(1) = \mathbb{k}$  in arity 1 and is trivial otherwise.

In general, we tacitely assume that an operad  $\mathbf{P}$  satisfies  $\mathbf{P}(0) = 0$  (we say that  $\mathbf{P}$  is non-unitary). In some cases, we may assume that an operad satisfies  $\mathbf{P}(1) = \mathbb{k}$  too. We say in this situation that  $\mathbf{P}$  is a connected operad. Cooperads  $\mathbf{D}$  are always assumed to satisfy the connectedness assumptions  $\mathbf{D}(0) = 0$  and  $\mathbf{D}(1) = \mathbb{k}$  in order to avoid difficulties with infinite sums in coproducts. We use the notation  $\mathcal{O}_0$  (respectively  $\mathcal{O}_1$ ) for the category of non-unitary (respectively connected) operads in dg-modules. We call  $\Sigma_*$ -object the structure, underlying an operad, formed by a collection  $M = \{M(r)\}_{r \in \mathbb{N}}$  where  $M(r)$  is a dg-modules equipped with an action of the symmetric group on  $r$  letters  $\Sigma_r$ .

A connected operad inherits a canonical augmentation  $\epsilon : \mathbf{P} \rightarrow \mathbf{l}$  which is the identity in arity  $r = 1$  and is trivial otherwise. We adopt the notation  $\tilde{\mathbf{P}}$  for the augmentation ideal of any operad  $\mathbf{P}$  equipped with an augmentation  $\epsilon : \mathbf{P} \rightarrow \mathbf{l}$ . We have in the connected case  $\tilde{\mathbf{P}}(0) = \tilde{\mathbf{P}}(1) = 0$  and  $\tilde{\mathbf{P}}(r) = \mathbf{P}(r)$  for all  $r \geq 2$ . Any

connected cooperad  $D$  has a coaugmentation  $\eta : I \rightarrow D$  and a coaugmentation coideal  $\tilde{D}$  defined like the augmentation and the augmentation ideal of a connected operad.

*0.3. Suspensions.* — The suspension of a dg-module  $C$  is the dg-module  $\Sigma C$  defined by the tensor product  $\Sigma C = \mathbb{k}[1] \otimes C$  where, for all  $d \in \mathbb{Z}$ , we adopt the notation  $\mathbb{k}[d]$  for the free graded  $\mathbb{k}$ -module of rank 1 concentrated in degree  $d$ .

The operadic suspension of an operad  $P$ , is an operad  $\Lambda P$  characterized by the commutation relation  $\Lambda P(\Sigma C) = \Sigma P(C)$  at the level of free algebras. Basically, the operad  $\Lambda P$  is defined arity-wise by the tensor products  $\Lambda P(n) = \mathbb{k}[1-n] \otimes P(n)^\pm$  where the notation  $\pm$  refers to a twist of the natural  $\Sigma_n$ -action on  $P(n)$  by the signature of permutations. We have an operadic suspension operation defined in the same way on cooperads.

Note that we can apply the suspension of dg-modules arity-wise to any operad  $P$  (or cooperad) to produce a  $\Sigma_*$ -object  $\Sigma P$  such that  $\Sigma P(r) = \mathbb{k}[1] \otimes P(r)$ , but this suspended  $\Sigma_*$ -object does not inherit an operad structure.

*0.4. Algebra and coalgebra categories.* — The category of algebras in dg-modules associated to an operad  $P$  is denoted by  ${}_P\mathcal{C}$ . The free  $P$ -algebra associated to a dg-module  $C$  is denoted by  $P(C)$ . Recall simply that this free  $P$ -algebra is defined by the dg-module of generalized symmetric tensors  $P(C) = \bigoplus_{r=0}^{\infty} (P(r) \otimes C^{\otimes r})_{\Sigma_r}$ , where the  $\Sigma_r$ -quotient identifies tensor permutations with the action of permutations on  $P(r)$ . Note that we can remove the 0 term of this expansion since we assume  $P(0) = 0$ . We use the notation  $p(x_1, \dots, x_r)$  where  $p \in P(r)$ ,  $x_1 \otimes \dots \otimes x_r \in A^{\otimes r}$ , to represent the element of  $P(C)$  defined by the tensor  $p \otimes (x_1 \otimes \dots \otimes x_r)$ . We may also use the graphical representation

$$(2) \quad \begin{array}{c} x_1 \cdots \cdots x_r \\ \swarrow \quad \quad \searrow \\ \cdots \quad \quad \cdots \\ \boxed{p} \\ \downarrow \\ \cdots \quad \quad \cdots \\ 0 \end{array}$$

for these elements  $p(x_1, \dots, x_r) \in P(C)$ .

In the case of a cooperad  $D$ , we use the notation  $D(C)$  for the connected cofree  $D$ -coalgebra such that  $D(C) = \bigoplus_{r=0}^{\infty} (D(r) \otimes C^{\otimes r})_{\Sigma_r}$ . In this definition, we take the same generalized symmetric tensor expansion as a free algebra over an operad for the underlying functor of a cofree coalgebra over a cooperad and therefore, we can safely adopt the functional and graphical representation of elements of free algebras of operads to denote the elements of  $D(C)$ . Note that  $D(C)$  does not agree with the standard cofree connected coalgebras of the literature in usual cases (like cocommutative coalgebras) precisely because we take coinvariants instead of invariants, but we generally apply the definition of  $D(C)$  to cooperads  $D$  equipped with a free  $\Sigma_*$ -module structure so that the result is the same if we replace coinvariants by invariants in the definition of  $D(C)$ .

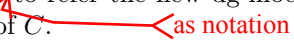
The coaugmentation morphism of a cooperad yields an embedding  $\eta : C \hookrightarrow D(C)$  which identifies the dg-module  $C$  with a summand of  $D(C)$ . Similarly, if  $P$  is an augmented operad, then we have a split embedding  $\eta : C \hookrightarrow P(C)$ , yielded by the

operad unit of  $P$ . The morphism  $\epsilon : P(C) \rightarrow C$  induced by the augmentation of  $P$  gives a canonical retraction of  $\eta : C \hookrightarrow P(C)$  so that  $C$  is canonically identified with a summand of  $P(C)$ .

*0.5. Model structures.* — The category of dg-modules  $\mathcal{C}$  is equipped with its standard model structure in which the weak-equivalences are the morphisms which induce an isomorphism in homology, the fibrations are the degreewise surjections (see [15, §2.3]).

The category of non-unitary operads  $\mathcal{O}_0$  inherits a full model structure from dg-modules so that a morphism  $f : P \rightarrow Q$  is a weak-equivalence (respectively, a fibration) if  $f$  forms a weak-equivalence (respectively, fibration) of dg-modules  $f : P(r) \rightarrow Q(r)$  in each arity  $r \in \mathbb{N}$ . The cofibrations are characterized by the right-lifting-property with respect to acyclic fibrations. The category of  $\Sigma_*$ -objects in dg-modules also inherits a full model structure with the same definition for the class of weak-equivalences (respectively, fibrations). Any operad cofibration with a cofibrant operad as domain defines a cofibration of  $\Sigma_*$ -objects, but the converse implication does not hold. Therefore we say that an operad  $P$  is  $\Sigma_*$ -cofibrant when its unit morphism  $\eta : I \rightarrow P$  defines a cofibration in the category of  $\Sigma_*$ -objects.

The category of algebras over a  $\Sigma_*$ -cofibrant operad  $P$  inherits a semi-model structure in the sense that the axioms of model categories are satisfied for  $P$ -algebras only when we restrict the applications of the right lifting property of (acyclic) cofibrations and of the factorization axiom to morphisms with a cofibrant domain. This is enough for the usual constructions of homotopical algebra. Naturally the weak-equivalences (respectively, fibrations) of  $P$ -algebras are the morphisms of  $P$ -algebras  $f : A \rightarrow B$  which define a weak-equivalence (respectively, a fibration) in the category of dg-modules.

*0.6. Twisted objects.* — In certain constructions, a homomorphism  $\partial \in \text{Hom}_{\mathcal{C}}(C, C)$  of degree  $-1$  is added to the internal differential of a dg-module  $C$  to produce a new dg-module, with the same underlying graded  $k$ -module as  $C$ , but the map  $\delta + \partial : C \rightarrow C$  as differential. Since we have  $\delta^2 = 0$ , the relation of differential  $(\delta + \partial)^2 = 0$  amounts to the equation  $\delta(\partial) + \partial^2 = 0$  in  $\text{Hom}_{\mathcal{C}}(C, C)$ . In this situation, we say that  $\partial$  is a twisting homomorphism and we ~~just~~ use the pair  $(C, \partial)$  to refer the new dg-module defined by the addition of  $\partial$  to the internal differential of  $C$ . 

When we apply this construction to an algebra  $A$  over an operad  $P$ , we assume that the twisting homomorphism satisfies the derivation relation  $\partial(p(a_1, \dots, a_r)) = \sum_{i=1}^r p(a_1, \dots, \partial(a_i), \dots, a_r)$  to ensure that the twisted dg-module  $(A, \partial)$  inherits a  $P$ -algebra structure from  $A$ . We then say that  $\partial : A \rightarrow A$  is a twisting derivation. We call quasi-free  $P$ -algebra the twisted  $P$ -algebras  $(A, \partial)$  associated to a free  $P$ -algebra  $A = P(C)$ . We have a one-one correspondence between derivations on free  $P$ -algebras  $\partial : P(C) \rightarrow P(C)$  and homomorphisms  $\gamma : C \rightarrow P(C)$ . We use the notation  $\partial_\gamma$  for the derivation associated to  $\gamma$ . In one direction, we have  $\gamma = \partial_\gamma|_C$ . In the other direction, the derivation  $\partial_\gamma : P(C) \rightarrow P(C)$  is determined from  $\gamma : C \rightarrow P(C)$  by the derivation formula  $\partial_\gamma(p(c_1, \dots, c_r)) = \sum_{i=1}^r p(c_1, \dots, \gamma(c_i), \dots, c_r)$ , for all  $p(c_1, \dots, c_r) \in P(C)$ . For a free  $P$ -algebra  $P(C)$ , a derivation  $\partial_\gamma : P(C) \rightarrow P(C)$  satisfies the equation of twisting homomorphisms if and only if the associated homomorphism  $\gamma : C \rightarrow P(C)$  satisfies the equation  $\delta(\gamma) + \partial_\gamma \cdot \gamma = 0$  in  $\text{Hom}_{\mathcal{C}}(C, P(C))$ .

We also apply the definition of twisted dg-modules to coalgebras over cooperads. We have a notion of quasi-cofree D-coalgebra, dual to the notion of quasi-free algebra over an operad, consisting of a twisted dg-module  $(D, \partial)$  such that  $D = D(C)$  is a cofree D-coalgebra and  $\partial = \partial_\alpha : D(C) \rightarrow D(C)$  is a D-coalgebra coderivation uniquely determined by a homomorphism  $\alpha : D(C) \rightarrow C$  such that  $\delta(\alpha) + \alpha \cdot \partial_\alpha = 0$  in  $\text{Hom}_C(D(C), C)$ . In the context of coalgebras, we determine the twisting coderivation  $\partial_\alpha$  associated to a homomorphism  $\alpha$  by a graphical expression of the form:

$$(3) \quad \partial_\alpha \left\{ \begin{array}{c} x_1 \cdots \cdots x_r \\ \vdots \\ \downarrow \\ c \\ \downarrow \\ 0 \end{array} \right\} = \sum_i \pm \left\{ \begin{array}{c} x_1 \cdots \cdots \alpha(x_i) \cdots \cdots x_r \\ \vdots \\ \downarrow \\ c \\ \downarrow \\ 0 \end{array} \right\} + \sum_{\substack{\tau \in \Theta_2(r) \\ \rho_\tau(c)}} \pm \left\{ \begin{array}{c} \alpha \left\{ \begin{array}{c} x_* \cdots \cdots x_* \\ \vdots \\ \downarrow \\ c_* \\ \downarrow \\ 0 \end{array} \right\} \cdots \cdots x_* \\ \vdots \\ \downarrow \\ c_* \\ \downarrow \\ 0 \end{array} \right\}.$$

The notation  $\rho_\tau(c)$  refers to a cooperadic coproduct along a tree  $\tau$  of an element  $c \in D(r)$  and the expressions  $c_*$  inside the sum refer to the factors of this cooperadic coproduct (see [6, Proposition 4.1.3] for details). The notation  $\Theta_2(r)$  refers to the category of trees with two vertices. The second sum ranges over all trees  $\tau \in \Theta_2(r)$  and, for each  $\tau \in \Theta_2(r)$ , over all terms of the expansion of  $\rho_\tau(c)$ . In the first sum, we just consider the restriction of  $\alpha$  to the summand  $C \subset D(C)$  and we sum up over all inputs of  $c \in D(r)$ .

### 1. Koszul duality and operadic homology

The purpose of this section is to review applications of operadic bar duality to the definition of homology theories for algebras over an operad.

The first reference addressing applications of homotopical algebra to algebras over operads in the setting of unbounded dg-modules over a ring is the article [13]. The definition of a homology theory with trivial coefficients, in terms of Quillen's homotopical algebra, was given earlier in [12] in the context of  $\mathbb{N}$ -graded dg-modules over a field of characteristic 0. The application of operadic bar duality to homotopical algebra of algebras over operads is explained in that reference in the same characteristic 0 setting. Briefly, the authors of [12] prove that the existence of a cofibrant model given by an operadic cobar construction  $B^c(D)$  for an operad  $P$  gives rise to an explicit cofibrant replacement functor on the category of  $P$ -algebras which can be used for homology computations.

In [6] we explain how to extend the applications of operadic bar duality to the context of unbounded dg-modules over a ring. More specifically, we prove that the functorial cofibrant replacement of algebras over an operad, defined [12], works in that



framework provided that we restrict ourself to  $\Sigma_*$ -cofibrant operads and to algebras which are cofibrant as dg-modules.

We review the definition of these cofibrant replacement functors very briefly before explaining the applications to the homology of algebras over operads in our setting. We explain at each step how the constructions apply to  $E_n$ -operads.

In this section (and in the next one), we assume that  $E_n$  is ~~the particular~~  $E_n$ -operad, arising from a filtration of the Barratt-Eccles operad [1], used in the proof of the Koszul duality result of [10]. This operad has, among others, the advantage of being free of finite rank over  $\mathbb{k}$  in each arity. Thus we can dualize the dg-modules  $E_n(r)$  without special care. Besides, the operad  $E_n$  is connected so that the collection of dual dg-modules  $E_n(r)^\vee$  inherits a well-defined cooperad structure.

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To begin with, we say a few words about the operadic cobar construction  $B^c(D)$ .

*1.1. On operadic cobar constructions.* — The operadic cobar construction is a functor which associates a connected operad  $B^c(D)$  to any cooperad  $D$ . We do not really need the explicit definition of  $B^c(D)$ . Just recall that  $B^c(D)$  is a cofibrant as an operad whenever the cooperad  $D$  is  $\Sigma_*$ -cofibrant and that any morphism  $\phi : B^c(D) \rightarrow P$  towards an operad  $P$  is fully determined by a homogeneous morphism  $\theta : D \rightarrow P$ , of degree  $-1$ , vanishing on  $D(1)$ , and satisfying the equation:

$$(4) \quad \delta(\theta) \left\{ \begin{array}{c} i_1 \quad \cdots \quad i_r \\ \downarrow \quad \quad \downarrow \\ c \\ \downarrow \\ 0 \end{array} \right\} = \sum_{\substack{\tau \in \Theta_2(I) \\ \rho_\tau(\gamma)}} \pm \lambda_* \left\{ \begin{array}{c} i_* \quad \cdots \quad i_* \\ \downarrow \quad \quad \downarrow \\ \theta(c_*) \\ \downarrow \\ \theta(c_*) \\ \downarrow \\ 0 \end{array} \right\},$$

for any  $c \in D(r)$ , where  $\lambda_*$  refers to the composition operation of the operad  $P$  and we adopt conventions similar to §0.6 to represent the expansion of the coproduct of a cooperad element  $c \in D(r)$ . We refer to [6, §3.7] for details on this matter.

We adopt the notation  $\phi = \phi_\theta$  for the morphism associated to  $\theta : D \rightarrow P$  and we refer to  $\theta$  as the twisting cochain associated to  $\phi_\theta$ .

For our  $E_n$ -operad, we have:

**Theorem 1.2** (see [10, Theorems A-B]). — *The operad  $E_n$  has a cofibrant model of the form  $Q_n = B^c(D_n)$ , where  $D_n = \Lambda^{-n} E_n^\vee$  is the  $n$ -fold operadic desuspension of the dual cooperad of  $E_n$  in dg-modules.*  $\square$

The proof of this assertion in [10] is divided ~~in~~ several steps. The first step involves the definition of a morphism  $\phi_n : B^c(D_n) \rightarrow C$  towards the operad of commutative algebras  $C$ . The weak-equivalence  $\psi_n : B^c(D_n) \xrightarrow{\sim} E_n$  is defined in a second step by a lifting process from the morphism  $\phi_n$ .

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*1.3. Cofibrant replacements arising from operadic cobar constructions.* — Suppose now we have a  $\Sigma_*$ -cofibrant connected operad  $P$  together with a cofibrant model of the form  $Q = B^c(D)$ , where  $D$  is a  $\Sigma_*$ -cofibrant operad. Let  $\phi_\theta : B^c(D) \xrightarrow{\sim} P$  be the augmentation of this cofibrant model.

To any  $\mathbf{P}$ -algebra  $A$ , we associate a (connected) quasi-cofree  $\mathbf{D}$ -coalgebra  $\Gamma_{\mathbf{P}}(A) = (\mathbf{D}(A), \partial_{\alpha})$  and a quasi-free  $\mathbf{P}$ -algebra of the form  $R_A = (\mathbf{P}(\Gamma_{\mathbf{P}}(A)), \partial_{\gamma})$ . The homomorphism  $\alpha$  which determines the twisting coderivation of  $\Gamma_{\mathbf{P}}(A)$  maps an element  $c(a_1, \dots, a_r) \in \mathbf{D}(A)$  to the evaluation of the operation  $\theta(c) \in \mathbf{P}(r)$  on  $a_1 \otimes \dots \otimes a_r \in A^{\otimes r}$ . The homomorphism  $\gamma$  which determines the twisting derivation of  $R_A$  is given by the composite

$$\Gamma_{\mathbf{P}}(A) \xrightarrow{\rho} \mathbf{D}(\Gamma_{\mathbf{P}}(A)) \xrightarrow{\theta(\Gamma_{\mathbf{P}}(A))} \mathbf{P}(\Gamma_{\mathbf{P}}(A)),$$

where  $\rho$  is yielded by the universal  $\mathbf{D}$ -coalgebra structure of the quasi-cofree  $\mathbf{D}$ -coalgebra  $\Gamma_{\mathbf{P}}(A)$ .

Then:

**Proposition 1.4** (see [12, §2] and [6, §4.2]). — *We have a weak-equivalence  $\epsilon : R_A \xrightarrow{\sim} A$  induced on generators  $\Gamma_{\mathbf{P}}(A) \subset R_A$  by the augmentation of the cofree  $\mathbf{D}$ -coalgebra  $\mathbf{D}(A) \rightarrow A$  and  $R_A$  defines a cofibrant replacement of  $A$  in  $\mathbf{P}$ -algebras provided that  $A$  is cofibrant as a dg-module.*  $\square$

We refer to [12] for the version of this statement in the context of  $\mathbf{N}$ -graded dg-modules over a field characteristic 0 and to [6] for the proof in our setting.

In [10, §1.3], we check that the cooperad  $\mathbf{D}_n = \Lambda^{-n} \mathbf{E}_n^{\vee}$  associated to our  $E_n$ -operad  $\mathbf{E}_n$  is  $\Sigma_*$ -cofibrant (like the operad  $\mathbf{E}_n$  itself). Hence the result of Proposition 1.4 holds for algebras over our  $E_n$ -operad.

**1.5. The cofibrant replacement functor and endomorphism operads.** — Recall that the action of an operad  $\mathbf{P}$  on an algebra  $A$  is determined by an operad morphism  $\nabla : \mathbf{P} \rightarrow \text{End}_A$ , where  $\text{End}_A$ , the endomorphism operad of  $A$ , is the operad such that  $\text{End}_A(r) = \text{Hom}_{\mathcal{C}}(A^{\otimes r}, A)$ . The evaluation of an operation  $p \in \mathbf{P}(r)$  on a tensor  $a_1 \otimes \dots \otimes a_r \in A^{\otimes r}$  is defined by the evaluation of the map  $\nabla(p) \in \text{Hom}_{\mathcal{C}}(A^{\otimes r}, A)$  on  $a_1 \otimes \dots \otimes a_r$ .

Observe that the homomorphism  $\alpha : \mathbf{D}(A) \rightarrow A$  in the definition of  $\Gamma_{\mathbf{P}}(A)$  is, by adjunction, equivalent to the homomorphism  $\alpha^{\#} : \mathbf{D} \rightarrow \text{End}_A$  such that  $\phi_{\alpha^{\#}}$  is the composite

$$B^c(\mathbf{D}) \xrightarrow{\phi_{\theta}} \mathbf{P} \xrightarrow{\nabla_A} \text{End}_A$$

giving the action of  $B^c(\mathbf{D})$  on  $A$  through  $\phi_{\theta}$ . Hence the twisting derivation of  $\Gamma_{\mathbf{P}}(A)$  is fully determined by the restriction of the  $\mathbf{P}$ -algebra structure of  $A$  to  $B^c(\mathbf{D})$ , and not by the  $\mathbf{P}$ -algebra structure itself.

In §3, we apply this observation to the operad  $\mathbf{E}_n$  because, in the case  $A = \bar{N}^*(S^{n-m})$ , we have a good characterization of the morphism  $\phi_{\alpha^{\#}} : B^c(\mathbf{D}_n) \rightarrow \text{End}_{\bar{N}^*(S^{n-m})}$ .

**1.6. Operadic homology.** — Let  $\mathbf{P}$  be an augmented operad. The indecomposable quotient of  $\mathbf{P}$ -algebra  $A$  is defined as the quotient dg-module of  $A$  under the image of operations  $p : A^{\otimes r} \rightarrow A$ , where  $p$  ranges over the augmentation ideal of  $\mathbf{P}$ . Thus:

$$\text{Indec}_{\mathbf{P}} A = A / \text{Im}(\tilde{\mathbf{P}}(A) \rightarrow A).$$

For a  $\Sigma_*$ -cofibrant augmented operad  $P$ , we have a homology theory  $H_*^P(-)$  defined by the formula

$$H_*^P(A) = H_*(\text{Indec}_P R_A)$$

for any  $P$ -algebra  $A$ , where  $R_A$  is a cofibrant replacement of  $A$ .

In the case of a quasi-free  $P$ -algebra, we have an identity  $\text{Indec}_P(P(C), \partial_\gamma) = C$ . Thus, when we apply the definition of the homology  $H_*^P(A)$  to the functorial cofibrant replacement  $R_A = (P(\Gamma_P(A)), \partial_\alpha)$ , we obtain the well-known assertion:

**Proposition 1.7.** — *Provided that the  $P$ -algebra  $A$  is cofibrant as a dg-module, we have the relation  $H_*^P(A) = H_*(\Gamma_P(A)) = H_*(D(A), \partial_\alpha)$ .*  $\square$

In the special case  $P = E_n$ , this relation gives the identity:

$$H_*^{E_n}(A) = H_*(D_n(A), \partial_\alpha) = H_*(\Lambda^{-n} E_n^\vee(A), \partial_\alpha).$$

Before applying this construction to the cochain algebra of a sphere, we go back to the general study of coalgebras  $D(C)$ .

## 2. Free complete algebras and duality

Throughout the paper, we adopt the notation  $\mathbb{k}[d]$  for a dg-module of rank one concentrated in (lower) degree  $d$ . The reduced normalized cochain complex of a sphere  $\bar{N}^*(S^d)$  is an instance of dg-module of this form. Since upper gradings are changed to opposite lower gradings, we obtain precisely  $\bar{N}^*(S^d) = \mathbb{k}[-d]$ .

The purpose of this section is to study coderivations of cofree coalgebras  $D(C)$  over a dg-module  $C = \mathbb{k}[m]$  with the aim to determine, in the next section, the complex  $(\Lambda^{-n} E_n^\vee(A), \partial_\alpha)$  associated to  $A = \bar{N}^*(S^{n-m})$ . To make our description more conceptual, the idea, explained in the introduction, is to define the complex in a dual setting of generalized power series algebras rather than in a coalgebra setting. Therefore we study the duality process first.

From now on, we assume, if not explicitly recalled, that  $x$  is a variable of degree  $-m$ .

**2.1. The free complete algebra on one variable over an operad.** — In our presentation, we consider the free  $P$ -algebra  $P(x)$  on a variable  $x$ , which can be defined as the free  $P$ -algebra  $P(\mathbb{k}x)$  on the free dg-module of rank one spanned by  $x$ . By definition, we have an identity  $P(x) = \bigoplus_{r=0}^\infty (P(r) \otimes \mathbb{k}x^{\otimes r})_{\Sigma_r}$ . To define the free complete  $P$ -algebra  $\hat{P}(x)$ , we just replace the sum in the expansion of the free  $P$ -algebra by a product (formed within the category of dg-modules):  $\hat{P}(x) = \prod_{r=0}^\infty (P(r) \otimes \mathbb{k}x^{\otimes r})_{\Sigma_r}$ . The free complete  $P$ -algebra is equipped with the topology defined by the nested sequence of dg-submodules  $\hat{P}^{(>s)}(x) = \prod_{r=s+1}^\infty (P(r) \otimes \mathbb{k}x^{\otimes r})_{\Sigma_r}$ .

**2.2. Duality.** — By [5, Proposition 1.2.18], the dual in dg-modules of a connected cooperad  $D$  always forms a connected operad. Now, if  $D$  is equipped with a free  $\Sigma_*$ -structure, then the dual of the connected cofree coalgebra  $D(C)$  on  $C = \mathbb{k}[m]$  can be identified with a complete free algebra  $\hat{P}(x)$  over the operad  $P = D^\vee$ , where  $x$  is a variable of degree  $-m$ , because we have in this case  $((D(r) \otimes \mathbb{k}[m]^{\otimes r})_{\Sigma_r})^\vee \simeq (D(r)^\vee \otimes \mathbb{k}[-m]^{\otimes r})^{\Sigma_r} \simeq (D(r)^\vee \otimes \mathbb{k}[-m]^{\otimes r})_{\Sigma_r}$  and the duality transforms the sum

into a product. In the sequel, we are rather interested in a converse duality operation, from free complete algebras over operads to coalgebras over cooperads.

The dual of a connected operad  $P^\vee$  does not inherit a cooperad structure in general, because we need a map  $(P(r) \otimes P(s))^\vee \rightarrow P(r)^\vee \otimes P(s)^\vee$  to separate the factors of the coproduct of  $P^\vee$ , but this condition is fulfilled when each dg-module  $P(r)$  is free locally of finite rank over  $\mathbb{k}$  and  $D = P^\vee$  does form a cooperad in that situation.

In the sequel, we apply the duality of dg-modules to the free complete  $P$ -algebra  $\widehat{P}(x)$ . In this case, the continuous dual with respect to the topology of  $\widehat{P}(x)$  is more natural to consider and the notation  $\widehat{P}(x)^\vee$  is used to refer to it. Thus, we have by definition  $\widehat{P}(x)^\vee = \text{colim}_s \{\widehat{P}(x)/\widehat{P}^{(>s)}(x)\}^\vee$ .

Besides, we will always assume that  $P$  is equipped with a free  $\Sigma_*$ -structure when we apply the duality to  $\widehat{P}(x)$ . In that setting, we obtain:

**Proposition 2.3.** — *Assume again that  $x$  is a homogeneous variable of degree  $-m$ . If the operad  $P$  is equipped with a free  $\Sigma_*$ -structure and is locally a free module of finite rank over  $\mathbb{k}$ , then we have an isomorphism  $\widehat{P}(x)^\vee \simeq D(\mathbb{k}[m])$ , where  $D = P^\vee$  is the dual cooperad of  $P$ .*  $\square$

Let  $\gamma^\sharp \in \widehat{P}(x)^\vee$  denote the continuous homomorphism associated to an element  $\gamma = c(x^\vee, \dots, x^\vee) \in D(\mathbb{k}[m])$ ,  $c \in D(r)$ , where we use the notation  $x^\vee$  for the canonical generator of the  $\mathbb{k}$ -module  $\mathbb{k}[m]$ . This homomorphism can be defined termwise by the relation  $\gamma^\sharp(p(x, \dots, x)) = 0$ , for  $p \in P(s)$ ,  $s \neq r$ , and the duality formula

$$\gamma^\sharp(p(x, \dots, x)) = \sum_{w \in \Sigma_r} c(w \cdot p),$$

for  $p \in P(r)$ .

The relation  $\widehat{P}(x)^\vee \simeq D(\mathbb{k}[m])$  is just converse to the relation  $D(\mathbb{k}[m])^\vee \simeq \widehat{P}(x)$  of §2.2 for  $D = P^\vee$ . By suspension, we deduce from these duality relations an isomorphism between the suspended dg-module  $\Sigma^{-m}\widehat{P}(x)$  and the dg-module of homomorphisms  $\alpha : D(\mathbb{k}[m]) \rightarrow \mathbb{k}[m]$ .

Recall that we also have a degree-preserving bijection between homomorphisms  $\alpha : D(\mathbb{k}[m]) \rightarrow \mathbb{k}[m]$  and coderivations  $\partial_\alpha : D(\mathbb{k}[m]) \rightarrow D(\mathbb{k}[m])$ . In fact this bijection defines an isomorphism of dg-modules when we equip the collection of coderivations  $\partial_\alpha : D(\mathbb{k}[m]) \rightarrow D(\mathbb{k}[m])$  with the natural dg-module structure coming from dg-module homomorphisms.

Now, on the side of the free complete  $P$ -algebra  $\widehat{P}(x)$ , we obtain:

**Proposition 2.4.** — *For any free complete  $P$ -algebra  $\widehat{P}(x)$  on one variable  $x$  of degree  $m$ , we have degree-preserving bijections between:*

- (a) *the elements  $\omega \in \widehat{P}(x)$ ,*
- (b) *the derivations of  $P$ -algebras  $\partial : \widehat{P}(x) \rightarrow \widehat{P}(x)$  which are continuous with respect to the topology of  $\widehat{P}(x)$ .*

This bijection defines an isomorphism of dg-modules too when we equip the collection of continuous derivations with the natural dg-module structure coming from dg-module homomorphisms.

In the sequel, we adopt the notation  $\partial_\omega : \widehat{\mathbf{P}}(x) \rightarrow \widehat{\mathbf{P}}(x)$  for the continuous derivation associated to an element  $\omega \in \widehat{\mathbf{P}}(x)$ .

*Proof.* — To check the proposition, we essentially observe that  $\partial_\omega$  is defined termwise by the formal identity

$$\partial_\omega(p(x, \dots, x)) = \sum_{i=1}^r p(x, \dots, \omega, \dots, x)$$

extending the formula of §0.6. The sum runs over all inputs of the operation  $p \in \mathbf{P}(r)$ . Then we use the continuity of the derivation to identify  $\partial_\omega$  on the whole dg-module  $\widehat{\mathbf{P}}(x)$ . In the converse direction, we simply retrieve the element  $\omega$  associated to a derivation  $\partial_\omega$  by the identity  $\omega = \partial_\omega(x)$ .  $\square$

From the explicit construction of the derivation  $\partial_\omega : \widehat{\mathbf{P}}(x) \rightarrow \widehat{\mathbf{P}}(x)$ , we obtain:

**Proposition 2.5.** — *Let  $\omega \in \widehat{\mathbf{P}}(x)$ . Let  $\omega^\sharp : \mathbf{D}(\mathbb{k}[m]) \rightarrow \mathbb{k}[m]$  be the homomorphism associated to  $\omega$  by the duality isomorphism  $\widehat{\mathbf{P}}(x) \simeq \mathbf{D}(\mathbb{k}[m])^\vee$  of §2.2.*

*The adjoint homomorphism of the derivation  $\partial_\omega : \widehat{\mathbf{P}}(x) \rightarrow \widehat{\mathbf{P}}(x)$  corresponds under the duality isomorphism  $\widehat{\mathbf{P}}(x)^\vee \simeq \mathbf{D}(\mathbb{k}[m])$  to the coderivation  $\partial_{\omega^\sharp} : \mathbf{D}(\mathbb{k}[m]) \rightarrow \mathbf{D}(\mathbb{k}[m])$  associated to  $\omega^\sharp$ .*

*Proof.* — Formal from the expression of the derivation  $\partial_\omega$  in the proof of proposition 2.4 of the coderivation  $\partial_{\omega^\sharp}$  in §0.6, and from the expression of the duality isomorphisms.  $\square$

The construction of quasi-free algebras over an operad can be adapted to free complete algebras. In that situation, we consider twisted dg-modules of the form  $(\widehat{\mathbf{P}}(x), \partial_\omega)$  where the twisting morphism  $\partial_\omega$  is a continuous derivation of  $\widehat{\mathbf{P}}(x)$  associated to some element  $\omega \in \widehat{\mathbf{P}}(x)$ . As in the context of quasi-free algebras, the derivation relation implies that the equation of twisting homomorphisms  $\delta(\partial_\omega) + \partial_\omega^2 = 0$  is fulfilled for  $\partial_\omega$  if and only if it holds on the generating element  $x$  of  $\widehat{\mathbf{P}}(x)$ . Because of the relation  $\omega = \partial_\omega(x)$ , this equation amounts to the identity  $\delta(\omega) + \partial_\omega(\omega) = 0$  in  $\widehat{\mathbf{P}}(x)$ .

The purpose of the next paragraphs is to explain that the composition structure considered in the introduction of this article models expressions of the form  $\partial_\alpha(\beta)$ , where  $\alpha, \beta \in \widehat{\mathbf{P}}(x)$ , and can be used to characterize twisting derivations on  $\widehat{\mathbf{P}}(x)$  (as well as twisting coderivations on the cofree coalgebra  $\mathbf{D}(\mathbb{k}[m])$  by the duality statement of proposition 2.5).

**2.6. The composition structure.** — To begin with, we review the definition of the composition structure considered in the introduction for a free complete algebra  $\widehat{\mathbf{P}}(x)$  over any operad  $\mathbf{P}$ . This definition is borrowed to [4] (we also refer to [16] for a variant of this composition structure).

One first defines the composite of elements of homogeneous order  $p(x, \dots, x)$ ,  $q(x, \dots, x) \in \widehat{\mathcal{P}}(x)$  by a sum of operadic substitution operations

$$p(x, \dots, x) \circ q(x, \dots, x) = \sum_{i=1}^r p(x, \dots, q(x, \dots, x), \dots, x)$$

running over all entries of  $p(x, \dots, x)$ . This termwise composition operation is continuous, and hence, extends to the completion  $\lim_{r,s} \{\widehat{\mathcal{P}}(x)/\widehat{\mathcal{P}}^{(>r)}(x) \otimes \widehat{\mathcal{P}}(x)/\widehat{\mathcal{P}}^{(>s)}(x)\}$ . For our purpose, we just consider the restriction of the obtained composition operation to the dg-module  $\widehat{\mathcal{P}}(x) \otimes \widehat{\mathcal{P}}(x) = \{\lim_r \widehat{\mathcal{P}}(x)/\widehat{\mathcal{P}}^{(>r)}(x)\} \otimes \{\lim_s \widehat{\mathcal{P}}(x)/\widehat{\mathcal{P}}^{(>s)}(x)\}$ , in order to obtain the operation:

$$\circ : \widehat{\mathcal{P}}(x) \otimes \widehat{\mathcal{P}}(x) \rightarrow \widehat{\mathcal{P}}(x).$$

Note that this composition operation decreases degrees by  $m$ , and becomes degree preserving only when we form  $m$ -desuspensions of the dg-module  $\widehat{\mathcal{P}}(x)$ .

This composition operation  $\circ$  is, after desuspension, an instance of a pre-Lie structure in the sense of [3], a product satisfying the identity

$$(a \circ b) \circ c - \pm(a \circ c) \circ b = \pm a \circ (b \circ c) - \pm a \circ (c \circ b).$$

The next lemma gives our motivation to introduce the pre-Lie composition operation:

**Lemma 2.7.** — *For any pair  $\alpha, \beta \in \widehat{\mathcal{P}}(x)$ , we have the identity  $\partial_\alpha(\beta) = \alpha \circ \beta$  in  $\widehat{\mathcal{P}}(x)$ .*

*Proof.* — This lemma is obvious from the explicit expression of the derivation  $\partial_\omega$  associated to any  $\omega \in \widehat{\mathcal{P}}(x)$  in the proof of proposition 2.4.  $\square$

This lemma implies:

**Proposition 2.8.** — *The derivation  $\partial_\omega : \widehat{\mathcal{P}}(x) \rightarrow \widehat{\mathcal{P}}(x)$  associated to any  $\omega \in \widehat{\mathcal{P}}(x)$  defines a twisting derivation on  $\widehat{\mathcal{P}}(x)$  if and only if we have the identity  $\delta(\omega) + \omega \circ \omega = 0$  in  $\widehat{\mathcal{P}}(x)$ .*  $\square$

In the next sections, we use the identity of Lemma 2.7 to express the derivation  $\partial_\omega$  associated by Proposition 2.4 to any element  $\omega \in \widehat{\mathcal{P}}(x)$ .

Before proceedings to the study of the chain complex  $C_*^{\mathbb{E}_n}(\bar{N}^*(S^{n-m}))$ , we record some applications of the pre-Lie composition structure to the definition of morphisms on the operad of  $L_\infty$ -algebras.

**2.9. On the operad of  $L_\infty$ -algebras.** — First of all, recall that this operad is defined by the cobar construction  $\mathcal{L}_\infty = \Lambda^{-1} B^c(\mathcal{C}^\vee)$ , where  $\mathcal{C}^\vee$  is the dual cooperad of the operad of commutative algebras. The operad  $\mathcal{L}_\infty$  comes equipped with a weak-equivalence  $\epsilon : \mathcal{L}_\infty \xrightarrow{\sim} \mathcal{L}$ , where  $\mathcal{L}$  is the operad of Lie algebras. Hence, we have an identity  $H_*(\mathcal{L}_\infty) = \mathcal{L}$  and the homology  $H_*(\mathcal{L}_\infty)$  is generated as an operad by an operation  $\lambda \in \mathcal{L}(2)$  satisfying the identities of a Lie bracket. This observation is used in §4. For the moment, we just want to record the following general proposition:

➤ The terminology is borrowed from M. Gerstenhaber : "The cohomology structure of an associative ring", Ann. of Math. (2) 78 (1963), 267–288. We refer to [3] for other historical references on pre-Lie structures.

**Proposition 2.10.** — *If the operad  $P$  is equipped with a free  $\Sigma_*$ -structure, then we have a bijection between:*

- (a) *the operad morphisms  $\phi : \Lambda L_\infty \rightarrow \Lambda^m P$ ;*
- (b) *and the elements  $\omega \in \widehat{P}(x)$ , of degree  $-1 - m$ , and such that the relation  $\delta(\omega) + \omega \circ \omega = 0$  holds in  $\widehat{P}(x)$ , where we still assume that  $x$  is a variable of degree  $-m$ .*

*Proof.* — The cobar construction commutes with operadic suspensions. Therefore we have an identity  $\Lambda L_\infty = B^c(C^\vee)$ .

Since  $C^\vee(r) = \mathbb{k}$  for all  $r > 0$ , the homomorphism  $\theta : C^\vee \rightarrow \Lambda^m P$  associated to  $\phi = \phi_\theta$  is equivalent to a collection of invariant elements in  $\Lambda^m P(r)^{\Sigma_r}$ . We use the assumption about the  $\Sigma_*$ -structure of  $P$  and the existence of an isomorphism  $P(r)^{\Sigma_r} \simeq P(r)_{\Sigma_r}$  to identify this collection with an element of the form  $\omega \in \widehat{P}(x)$ . We easily check that the equation of §1.1 for the twisting cochain  $\theta$  amounts to the equation  $\delta(\omega) + \omega \circ \omega = 0$  for the element  $\omega$  and the proposition follows.  $\square$

### 3. The Koszul complex on the cochain algebras of spheres

The aim of this section is to make explicit the complex  $(D_n(A), \partial_\alpha)$  computing  $H_*^{E_n}(A)$  for the cochain algebra of a sphere  $A = \bar{N}^*(S^{n-m})$ .

We still consider the ~~particular~~  $E_n$ -operads  $E_n$  arising from a filtration

specific

$$(5) \quad E_1 \subset \cdots \subset E_{n-1} \subset E_n \subset \cdots \subset E_\infty = E$$

of the chain Barratt-Eccles operad  $E$ . We do not need to review the definition of these operads  $E_n$  and of the Barratt-Eccles operad  $E$ . We are simply going to explain a representation of the  $E_n$ -algebra structure of  $\bar{N}^*(S^{n-m})$  in terms of realizations, for these operads  $E_n$ , of intrinsic homotopical structures of  $E_n$ -operads. This representation will give the form of the complex  $(D_n(A), \partial_\alpha)$  associated to  $A = \bar{N}^*(S^{n-m})$ .

We review the definition of the  $E_\infty$ -algebra structure of  $\bar{N}^*(S^{n-m})$  first, before explaining applications of the Koszul duality result of [10] to the cochain algebra  $\bar{N}^*(S^{n-m})$ .

*3.1. The cochain algebra of spheres.* — Recall that the (reduced normalized) cochain complex of a  $d$ -sphere  $S^d$  is identified with the free  $\mathbb{k}$ -module of rank 1 concentrated in (lower) degree  $-d$ :

$$\bar{N}^*(S^d) = \mathbb{k}[-d].$$

The associated endomorphism operad  $End_{\bar{N}^*(S^d)}$  is identified with the  $d$ -fold operadic desuspension of the commutative operad  $C$  because we have identities:

$$End_{\bar{N}^*(S^d)}(r) = \text{Hom}_C(\mathbb{k}[-d]^{\otimes r}, \mathbb{k}[-d]) = \mathbb{k}[rd - d] = \Lambda^{-d} C(r).$$

Hence, the  $E_\infty$ -structure of  $\bar{N}^*(S^d)$  is determined by an operad morphism  $\nabla_d : E_\infty \rightarrow \Lambda^{-d} C$ .

The work [2] gives an explicit representation of this morphism for the chain Barratt-Eccles operad. For our purpose, we use that  $\nabla_d$  is identified with a composite

$$E_\infty \xrightarrow{\sigma} \Lambda^{-1} E_\infty \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \Lambda^{-m} E_\infty \xrightarrow{\epsilon} \Lambda^{-m} C,$$

where  $\epsilon$  denotes the augmentation of the Barratt-Eccles operad (to which we apply the operadic suspension functor) while  $\sigma$  denotes a new operad morphism  $\sigma : \mathbf{E}_\infty \rightarrow \Lambda^{-1} \mathbf{E}_\infty$  (to which we also apply the operadic suspension) discovered in [2] and defined by an explicit formula.

In [10, Observation 0.1.10], we observe that  $\sigma : \mathbf{E}_\infty \rightarrow \Lambda^{-1} \mathbf{E}_\infty$  has restrictions giving operad morphisms  $\sigma : \mathbf{E}_n \rightarrow \Lambda^{-1} \mathbf{E}_{n-1}$ , for each  $n \geq 2$ , and we prove:

**Theorem 3.2** (see [10, Theorems A-B]). — *The weak-equivalences  $\psi_n : B^c(\mathbf{D}_n) \xrightarrow{\sim} \mathbf{E}_n$ , already considered in Theorem 1.2, fit a commutative diagram*

$$(6) \quad \begin{array}{ccc} B^c(\mathbf{D}_{n-1}) & \xrightarrow{\sigma^*} & B^c(\mathbf{D}_n) \\ \psi_{n-1} \downarrow \sim & & \psi_n \downarrow \sim \\ \mathbf{E}_{n-1} & \xrightarrow{\iota} & \mathbf{E}_n \end{array},$$

where  $\iota : \mathbf{E}_{n-1} \hookrightarrow \mathbf{E}_n$  is the embedding morphism and  $\sigma^*$  is the morphism associated to  $\sigma : \mathbf{E}_n \rightarrow \Lambda^{-1} \mathbf{E}_{n-1}$  by functoriality of the construction  $B^c(\mathbf{D}_m) = B^c(\Lambda^{-m} \mathbf{E}_m^\vee)$ .

We recall, just after Theorem 1.2, that the weak-equivalence  $\psi_m : B^c(\mathbf{D}_m) \xrightarrow{\sim} \mathbf{E}_m$  arises as a lifting

$$\begin{array}{ccc} & \mathbf{E}_m & \hookrightarrow \mathbf{E} \\ \psi_m \nearrow & & \downarrow \epsilon \\ B^c(\mathbf{D}_m) & \xrightarrow{\phi_m} & \mathbf{C} \end{array}$$

of a certain morphism  $\phi_m$ . The notation  $\epsilon$  refers again to the augmentation of the Barratt-Eccles operad.

By cobar duality, we have an equivalence  $(6) \iff (7)$ , where:

$$(7) \quad \begin{array}{ccc} B^c(\mathbf{E}_n^\vee) & \xrightarrow{\iota^*} & B^c(\mathbf{E}_{n-1}^\vee) \\ \psi_n^\# \downarrow \sim & & \psi_{n-1}^\# \downarrow \sim \\ \Lambda^n \mathbf{E}_n & \xrightarrow{\sigma} & \Lambda^{n-1} \mathbf{E}_{n-1} \end{array}.$$

The notation  $\iota^*$  now refers to the morphism associated to the embedding  $\iota : \mathbf{E}_{n-1} \hookrightarrow \mathbf{E}_n$  by functoriality of the construction  $B^c((-)^\vee)$ . From this result, we deduce:



**Theorem 3.3** (see [10, Theorem C]). — For every  $n > m$ , we have a commutative diagram

$$\begin{array}{ccccc}
 B^c(\Lambda^{-n} E_n^\vee) & \xrightarrow{\sim} & E_n & \hookrightarrow & E \\
 \downarrow \iota^* & & \downarrow \sigma & & \downarrow \sigma \\
 \vdots & & \vdots & & \vdots \\
 \downarrow \iota^* & & \downarrow \sigma & & \downarrow \sigma \\
 B^c(\Lambda^{-n} E_m^\vee) & \xrightarrow{\simeq} \Lambda^{m-n} B^c(\Lambda^{-m} E_m^\vee) \xrightarrow{\sim} \Lambda^{m-n} E_m & \hookrightarrow & \Lambda^{m-n} E & \\
 & \searrow \Lambda^{m-n} \phi'_m & & \downarrow \sim & \\
 & & & \Lambda^{m-n} C = \text{End}_{\bar{N}^*(S^{n-m})} & 
 \end{array}$$

giving the action of the  $E_n$ -operad  $B^c(\Lambda^{-n} E_n^\vee)$  on  $\bar{N}^*(S^{n-m})$ , where  $\phi'_m$  is a morphism homotopic to  $\phi_m$  in the model category of operads.

Note: one can easily observe that the cobar construction commutes with operadic suspensions.

To determine the form of the complex  $C_{E_n}^*(\bar{N}^*(S^{n-m}))$ , we essentially use the observation of §1.5 and the representation of the morphism  $B^c(D_n) \rightarrow \text{End}_{\bar{N}^*(S^{n-m})}$  supplied by this theorem.

By [6, Theorem 5.2.2] or [8, Theorem C], any  $\mathbb{Q}$ -algebras  $(A, \phi^0)$  and  $(A, \phi^1)$  having the same underlying dg-module  $A$  but a different  $\mathbb{Q}$ -structure determined by morphisms  $\phi^0, \phi^1 : \mathbb{Q} \rightarrow \text{End}_A$  are connected by a chain of weak-equivalences of  $\mathbb{Q}$ -algebras when the morphisms  $\phi^0, \phi^1$  are homotopic in the category of operads. Hence, in our study of the cochain algebra  $\bar{N}^*(S^{n-m})$ , any choice of morphism  $B^c(D_n) \rightarrow \text{End}_{\bar{N}^*(S^{n-m})}$  in the homotopy class of the composite

$$B^c(D_n) = B^c(\Lambda^{-n} E_n^\vee) \xrightarrow{\iota^*} B^c(\Lambda^{-n} E_m^\vee) \simeq \Lambda^{m-n} B^c(\Lambda^{-m} E_m^\vee) \xrightarrow{\phi'_m} \Lambda^{m-n} C$$

will give the right result. For that reason, we can replace the morphism  $\phi'_m$  in Theorem 3.3 by  $\phi_m$ , or any homotopy equivalent morphism  $\phi''_m$ . Besides, the construction of [10, §1] gives a morphism  $\phi_m$  well-characterized up to homotopy only.

Let us review the definition of these morphisms  $\phi_m$  before going further in our study.

**3.4. The augmentation on the cobar constructions.** — The morphisms  $\phi_m : B^c(D_m) \rightarrow C$  are, according to the recollections of §1.1, determined by homomorphisms of  $\Sigma_*$ -modules

$$D_m = \Lambda^{-m} E_m^\vee \xrightarrow{\theta_m} C$$

satisfying a certain equation. In [10, §§1.1.1-2], we observe that the definition of  $\theta_m$  amounts to the definition of a collection of elements  $\omega_m(r) \in E_m(r)_{\Sigma_r}$  of degree  $m(r-1)-1$ , or equivalently to an element  $\omega_m$  of degree  $-1-m$  in the completed

$E_m$ -algebra  $\widehat{E}_m(x) = \prod_{r=1}^{\infty} (E_m(r) \otimes \mathbb{k} x^{\otimes r})_{\Sigma_r}$ , where  $x$  is again a variable of degree  $-m$ . The element  $\omega_m(r)$  simply represents the term of order  $r$  of  $\omega_m \in \widehat{E}_m(x)$ .

Recall that  $C(r) = \mathbb{k}$  for all  $r > 0$ . The equivalence between  $\omega_m(r)$  and  $\theta_m$  is given by the adjunction relation  $\theta_m(\xi) = \sum_{w \in \Sigma_r} \xi(w \cdot \omega_m(r))$ , for any  $\xi \in \Lambda^{-m} E_m^{\vee}(r)$ , where  $\omega_m(r)$  is any representative of  $\omega_m(r)$  in  $E_m(r)$ . In the relation, we use the isomorphism  $E_m(r)_{\Sigma_r} \xrightarrow{\simeq} E_m(r)^{\Sigma_r}$  defined by the norm of the  $\Sigma_r$ -action (recall that  $\Sigma_r$  acts freely on  $E(r)$ ).

By inspection of [10, Lemma 1.1.2], we immediately see that the equation of the homomorphism  $\theta_m$  amounts to the equation

$$(8) \quad \delta(\omega_m) + \omega_m \circ \omega_m = 0$$

in term of the element  $\omega_m \in \widehat{E}_m(x)$ , where  $\circ : \widehat{E}_m(x) \otimes \widehat{E}_m(x) \rightarrow \widehat{E}_m(x)$  refers to the composition operation of §2.6.

Hence, we have by Proposition 2.8:

**Lemma 3.5.** — *The element  $\omega_m \in \widehat{E}_m(x)$  associated to the homomorphism  $\theta_m$  defining  $\phi_m$  determines, for each  $n \geq m$ , a twisting derivation  $\partial_m : \widehat{E}_n(x) \rightarrow \widehat{E}_n(x)$  such that  $\partial_m(\xi) = \xi \circ \omega_m$ , for any  $\xi \in \widehat{E}_n(x)$ .  $\square$*

Then we observe:

**Lemma 3.6.** — *The adjoint homomorphism of the derivation  $\partial_m : \widehat{E}_n(x) \rightarrow \widehat{E}_n(x)$  corresponds, under the isomorphisms*

$$\Sigma^{-n} \widehat{E}_n(x)^{\vee} \simeq \Sigma^{-n} E_n^{\vee}(\mathbb{k}[m]) \simeq \Lambda^{-n} E_n^{\vee}(\mathbb{k}[m-n]) = D_n(N^*(S^{n-m}))$$

*to the twisting coderivation  $\partial_{\alpha} : D_n(N^*(S^{n-m})) \rightarrow D_n(N^*(S^{n-m}))$  of the complex  $C_*^{E_n}(N^*(S^{n-m}))$  computing  $H_*^{E_n}(N^*(S^{n-m}))$ .*

*Proof.* — In §1.5, we observe that the homomorphism  $\alpha : D_n(\mathbb{k}[m-n]) \rightarrow \mathbb{k}[m-n]$  defining the coderivation of  $C_{E_n}^*(N^*(S^{n-m}))$  corresponds, by adjunction, to the twisting cochain  $D_n \rightarrow \text{End}_{\mathbb{k}[m-n]} = \text{End}_{N^*(S^{n-m})}$  determining the action of the operad  $B^c(D_n)$  on  $N^*(S^{n-m})$ . By Theorem 3.3, this twisting cochain is given by a composite:

$$D_n \xrightarrow{\iota^*} D_m \xrightarrow{\theta_m} \Lambda^{m-n} C = \text{End}_{\mathbb{k}[m-n]},$$

where  $\iota^*$  is, up to operadic suspension, the dual of the embedding morphism  $\iota : E_m \rightarrow E_n$ . If we apply the formula of  $\theta_m$  in terms of the element  $\omega_m \in \widehat{E}_m(x)$ , then we obtain the identity

$$\alpha(c(x^{\vee}, \dots, x^{\vee})) = \sum_{w \in \Sigma_r} c(w \cdot \omega_m(r))$$

for any  $c \in D_n(r) = E_n(r)^{\vee}$ , where we go back to the notation used around Proposition 2.3 for the elements of  $D_n(\mathbb{k}[m-n])$ . Thus we immediately see that  $\alpha$  agrees with the homomorphism associated to  $\omega_m \in \widehat{E}_m(x)$  in §§2.3-2.4, and the adjunction relation between the coderivation  $\partial_{\alpha}$  and the derivation  $\partial_m(\xi) = \xi \circ \omega_m$  associated to  $\omega_m$  follows from the assertion of Proposition 2.5.  $\square$

From this statement, we conclude:

**Theorem 3.7.** — We have an identity  $H_*^{\mathbb{E}_n}(\bar{N}^*(S^{n-m})) = \Sigma^{-n} H_*(\widehat{\mathbb{E}}_n(x)^\vee, \partial_m^\vee)$ , where we consider the twisting derivation  $\partial_m(\xi) = \xi \circ \omega_m$  of Lemma 2.4 and the continuous dual of the twisted complete  $\mathbb{E}_m$ -algebra  $(\widehat{\mathbb{E}}_n(x), \partial_m)$ .  $\square$

The theorem of the introduction follows from this result and:

**Theorem 3.8** (see [7, 9]). — Take  $\mathbb{k} = \mathbb{Z}$  as ground ring. For any space  $X$  whose homology  $H_*(X)$  is a degree-wise finitely generated  $\mathbb{Z}$ -module, we have an identity

$$H_*^{\mathbb{E}_n}(\bar{N}^*(X)) = \Sigma^{-n} H^*(\widehat{\Omega^n X}),$$

where  $\widehat{\Omega^n X}$  refers to the profinite completion of the  $n$ -fold loop space  $\Omega^n X$  (of which we take the continuous cohomology).  $\square$

We have explained around Theorem 3.3 that the result of Theorem 3.7 does not depend on the choice of the morphism  $\phi_m$  in a certain homotopy class because any such choice gives rise to weakly-equivalent models of the  $E_n$ -algebra underlying  $N^*(S^{n-m})$ . We prove in the next section that a wider homotopy invariance property holds for Theorem 3.7.

**3.9. Remark: the cohomological analogue of Theorem 3.7.** — We have a result similar to Lemma 3.6 for the complex  $C_{\mathbb{E}_n}^*(A, A)$  computing the natural cohomology theory  $H_{\mathbb{E}_n}^*(A, A)$  associated to  $E_n$ -operads. As a byproduct, we also have a cohomological analogue of the result of Theorem 3.7. At the complex level, we obtain an identity of the form

$$C_{\mathbb{E}_n}^*(\bar{N}^*(S^{n-m}), \bar{N}^*(S^{n-m})) = \Sigma^m (\widehat{\mathbb{E}}_m(x), ad_m),$$

where we equip the dg-module  $\widehat{\mathbb{E}}_m(x)$  with the twisting cochain such that  $ad_m(\xi) = \xi \circ \omega_m - \pm \omega_m \circ \xi$ .

#### 4. Homotopical invariance

In the previous section, we have used a filtration layer of the Barratt-Eccles operad as a preferred  $E_n$ -operad, but we mention in the introduction of this article that the result of our theorems holds for any choice of  $E_m$ -operad  $\mathbb{E}_m$  and for any choice of morphism  $\phi_m : B^c(\mathbb{D}_m) \rightarrow \mathbb{C}$ . The goal of this section is precisely to prove this wide homotopy invariance property.

Recall that  $H_*(\mathbb{E}_m)$  is identified with the operad of associative algebras  $\mathbf{A}$  for  $m = 1$ , with the  $m$ -Gerstenhaber operad  $\mathbf{G}_m$  for  $m > 1$ . The associative operad is generated as an operad by an associative product operation  $\mu \in \mathbf{A}(2)$ , of degree 0. The  $m$ -Gerstenhaber operad is generated by an associative and commutative product operation  $\mu \in \mathbf{G}_m(2)$  of degree 0 and by a Lie operation  $\lambda \in \mathbf{G}_m(2)$  of degree  $m - 1$  satisfying a graded version of the Poisson distribution relation.

For the moment, we still consider the  $E_m$ -operad  $\mathbb{E}_m$  of [10], arising from the filtration of the Barratt-Eccles operad. In this context, the morphisms  $\phi_m : B^c(\mathbb{D}_m) \rightarrow \mathbb{C}$  are characterized by the relation  $\phi_m(\mu) = \mu$  and  $\phi_m(\lambda) = 0$  in homology – where we use the isomorphism  $H_*(B^c(\mathbb{D}_m)) \simeq H_*(\mathbb{E}_m)$  (established in [10, §0.3]), because we have:

**Lemma 4.1** (see [11, Theorem B]). — *Any pair of morphisms  $\phi_m^\epsilon : B^c(D_m) \rightarrow C$ ,  $\epsilon = 0, 1$ , satisfying the relations  $\phi_m^\epsilon(\mu) = \mu$  and  $\phi_m^\epsilon(\lambda) = 0$  in homology are left homotopic in the model category of operads.*  $\square$

Note that the relation  $\phi_m(\lambda) = 0$  is obvious for  $m > 1$  since  $C$  is concentrated in degree 0. For the element  $\omega_m \in \widehat{E}_m(x)$  associated to  $\phi_m$ , the relation  $\phi_m(\mu) = \mu$  amounts to the identity  $[\omega_m(2)] = \mu$  in  $H_*(E_m(2)_{\Sigma_2})$ , where  $\omega_m(2)$  refers to the component of  $\omega_m$  of order 2. Recall that  $\omega_m(1) = 0$  and the relation  $\delta(\omega_m) + \omega_m \circ \omega_m = 0$  implies at order 2 that  $\omega_m(2)$  defines a cycle in  $E_m(2)_{\Sigma_2}$  (see [10, Lemma 1.1.2]).

Thus the lemma has an obvious interpretation in term of the element  $\omega_m$ .

Now we plan to extend the result of Theorem 3.7 to  $E_n$ -operads  $\Xi_n$  which are  $\Sigma_*$ -cofibrant but not necessarily finitely generated in all degree and for each arity. In that context, we consider the operad  $L_\infty$ , associated to the usual category of  $L_\infty$ -algebras, defined by the cobar construction  $L_\infty = \Lambda^{-1}B^c(C^\vee)$ , where  $C^\vee$  is the dual cooperad of the operad of commutative algebras. Recall that  $H_*(L_\infty)$  is isomorphic to the operad of Lie algebras  $L$  which is generated, as an operad, by an operation  $\lambda \in L(2)$  satisfying the identities of a Lie bracket (see §2.9).

In the next statement, we use the operadic suspension of this operad  $\Lambda L_\infty = B^c(C^\vee)$  for which the homology is generated, as an operad, by a Lie bracket operation  $\lambda \in L(2)$  of degree 2. Since we can not form the dual of  $\Xi_m$  when  $\Xi_m$  is not free of finite rank in all degree and for each arity, we can not use the relationship between twisting elements  $\omega_m \in \widehat{E}_m(x)$  and morphisms  $\phi_m : B^c(\Lambda^{-m}\Xi_m^\vee) \rightarrow C$ , but we still have:

**Proposition 4.2.** — *The definition of a morphism  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^m \Xi_m$ , amounts to the definition of an element  $\xi_m \in \widehat{\Xi}_m(x)$ , of degree  $-1 - m$ , vanishing at order 1, and satisfying  $\delta(\xi_m) + \xi_m \circ \xi_m = 0$  in  $\widehat{\Xi}_m(x)$ . Moreover, we have the identity  $\phi_m^\sharp(\lambda) = \lambda$  in homology if and only if  $[\xi_m(2)] = \mu$  in  $H_*(\Xi_m(2)_{\Sigma_2})$ , where  $\xi_m(2)$  represents the term of order 2 of  $\xi_m$ .*

*Proof.* — The first part of the proposition is established in Proposition 2.10. The second assertion of the proposition is an obvious consequence of the relation, established in the proof of Proposition 2.10, between the homomorphism  $\theta_m^\sharp$  which determines  $\phi_m^\sharp$  and the element  $\xi_m$ .  $\square$

The morphisms  $\phi_m^\sharp : \Lambda L_\infty \rightarrow \Lambda^m \Xi_m$  corresponds by bar duality to the morphisms  $\phi_m : B^c(\Lambda^{-m}\Xi_m^\vee) \rightarrow C$  of §3.4 whenever this correspondence makes sense (see [11]).

Suppose now we have an  $E_m$ -operad  $\Xi_m$  together with a morphism of the form of Proposition 4.2 and let  $\xi_m$  be the associated element in the free complete  $\Xi_m$ -algebra  $\widehat{\Xi}_m(x)$ . We aim at comparing the pair  $(\Xi_m, \xi_m)$  to a pair  $(E_m, \omega_m)$  formed from the Barratt-Eccles operad. We already know that the operads  $E_m$  and  $\Xi_m$  are connected by a chain of weak-equivalences:

$$E_m \xleftarrow{\phi} \Pi_m \xrightarrow{\psi} \Xi_m.$$

We can moreover assume that the morphism  $\phi$  and  $\psi$  are fibrations. We have then:

**Lemma 4.3.** — We have elements  $\pi_m \in \widehat{\Pi}_m(x)$  and  $\omega_m \in \widehat{E}_m(x)$  such that  $\phi(\pi_m) = \omega_m$  and  $\psi(\pi_m) = \xi_m$  and satisfying the identity  $[\pi_m(2)] = [\omega_m(2)] = \mu$  in homology, where as usual  $\pi_m(2)$  (respectively,  $\omega_m(2)$ ) denotes the term of order 2 of  $\pi_m$  (respectively,  $\omega_m$ ).

*Proof.* — The idea is to define the components  $\pi_m(r)$  of the element  $\pi_m \in \widehat{\Pi}_m(x)$  by induction on  $r$ . For this, we use that the equation  $\delta(\pi_m) + \pi_m \circ \pi_m = 0$  amounts at order  $r$  to an equation of the form:

$$(9) \quad \delta(\pi_m(r)) = \sum_{s+t=r-1} \left\{ \sum_{i=1}^r \pi_m(s) \circ_i \pi_m(t) \right\},$$

where we use the standard partial composite notation of the operadic substitution operation occurring in the definition of the composition product  $\circ : \widehat{\Pi}_m(x) \otimes \widehat{\Pi}_m(x) \rightarrow \widehat{\Pi}_m(x)$ . Thus, whenever appropriate elements  $\pi_m(s)$  such that  $\psi(\pi_m(s)) = \xi_m(s)$  are defined for  $s < r$ , the existence of  $\pi_m(r)$  amounts to the vanishing of an obstruction in  $H_*(\Pi_m(r)_{\Sigma_r})$ . Now this obstruction does vanish just because  $\psi$  induces an isomorphism in homology, preserves operadic structures, and carries (9) to the same equation for  $\xi_m$ . Hence we are done with the definition of  $\pi_m$ .

To define  $\omega_m$ , we just set  $\omega_m = \phi(\pi_m)$ .  $\square$

**Lemma 4.4.** — We have a chain of weak-equivalences

$$(\widehat{E}_m(x), \partial_m)^\vee \xleftarrow[\sim]{\phi} (\widehat{\Pi}_m(x), \partial_m)^\vee \xrightarrow[\sim]{\psi} (\widehat{\Xi}_m(x), \partial_m)^\vee$$

between the continuous duals of the chain complexes formed from the elements of Lemma 4.3.

*Proof.* — The commutation of  $\phi$  and  $\psi$  with operad structures readily implies that  $\phi$  and  $\psi$  induce dg-module morphisms between the twisted complexes. By the standard spectral sequence argument, we obtain that these dg-module morphisms are weak-equivalences since  $\phi$  and  $\psi$  induce weak-equivalences at the level of cofree coalgebras.  $\square$

Since the complex  $(\widehat{E}_n(x), \partial_m)$  formed from the Barratt-Eccles operad in Lemma 4.4 fits the construction of §3, we conclude:

**Theorem 4.5.** — Let  $\Xi_1 \subset \dots \subset \Xi_n \subset \dots$  be any nested collection of  $\Sigma_*$ -cofibrant operads equivalent to the nested sequence of the chain operads of little cubes. Suppose we have an operad morphism  $\phi_m^\sharp : \Lambda \mathbf{L}_\infty \rightarrow \Lambda^m \Xi_m$  satisfying  $\phi_m^\sharp(\lambda) = \lambda$  in homology and consider the equivalent element  $\xi_m \in \widehat{\Xi}_m(x)$  such that  $\delta(\xi_m) + \xi_m \circ \xi_m = 0$ .

Then the result of Theorem 3.7 holds for the complex  $(\widehat{\Xi}_n(x), \partial_m)$  formed from the element  $\xi_m \in \widehat{\Xi}_m(x)$ .  $\square$

In this theorem, we do not have to assume that the operad  $\Xi_m$  are finitely generated. In particular, we can apply the theorem to the  $E_n$ -operads considered in [17] formed by the semi-algebraic chain complex of the Fulton-MacPherson operad.

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